TORSION FREE MODULES OF NON-TWISTED AFFINE ALGEBRAS

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Abstract. It is shown that all torsion free modules for non-twisted Affine Algebras having a 1-dimensional weight space are the result of the natural construction of tensoring the Laurent polynomials with a torsion free module of the "underlying"simple finite dimensional Lie Algebra.These latter modules have been completely determined by Britten and Lemire [1].

1 Introduction. In the theory of finite dimensional simple Lie algebras over the complex numbers C, pointed torsion free modules played a central role in the classification [1] of infinite dimensional modules having a 1-dimensional one weight space. In this article, we classify the pointed torsion free modules for non-twisted affine Lie algebras.

We begin by presenting the construction of the non-twisted affine algebras as found in [5] and summarizing that portion of the finite dimensional theory which we require. We postpone the statement of our main result until Section 2. Our basic references are Humphreys [4] for finite dimensional Lie theory and Kac [5] for infinite dimensional Lie theory.

Let $L = \mathbf{C}[t^{-1}, t]$ be the algebra of Laurent polyonomials and let $\overset{0}{g}$ be a finite dimensional simple Lie algebra over C with a fixed Cartan subalgebra $\overset{0}{H}$. Let $L(\overset{0}{g}) = L \otimes \overset{0}{g}$ be the tensor product of L and $\overset{0}{g}$ over the complex numbers. Following Kac, we find that the non-twisted affine Lie algebras can be realized as

$$\hat{L}(\overset{0}{g}) = L(\overset{0}{g}) \oplus \mathbf{C}c \oplus \mathbf{C}d \tag{1.1}$$

where after fixing a non-degenerate symmetric bilinear C-valued form $(.|.)$, the multiplication is given by

$$\begin{aligned}(1.2)\quad & [t^k \otimes x + \nu c + \mu d, t^{k_1} \otimes y + \nu_1 c + \mu_1 d] \\ & \qquad = t^{k+k_1} \otimes [x, y] + \mu k_1 t^{k_1} \otimes y - \mu_1 k t^k \otimes x + k\delta_{k,-k_1}(x|y)c\end{aligned}$$

For a detailed discussion of (1.2) see [5,p.74]. The set $H = (1 \otimes \overset{0}{H}) \oplus \mathbf{C}c \oplus \mathbf{C}d$ is a Cartan subalgebra for $\hat{L}(\overset{0}{g})$.

In this paper we use the terms simple and irreducible interchangeably when speaking of modules.

DEFINITION. *A nonzero $\overset{0}{H}$-diagonalizable $\overset{0}{g}$-module $\overset{0}{V}$ is said to be* torsion free *provided that each root vector x acts in a $1-1$ manner on all weight spaces. Similarly, a nonzero*

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H-diagonalizable $\hat{L}(\overset{0}{g})$-module $\mathcal{V}$ is said to be **torsion free** *provided that, for all integers k and all root vectors $x \in \overset{0}{g}$, $t^k \otimes x$ acts in a $1-1$ manner on all weight spaces.*

DEFINITION. *An $\overset{0}{H}$-diagonalizable $\overset{0}{g}$-module $\overset{0}{\mathcal{V}}$ or an H-diagonalizable $\hat{L}(\overset{0}{g})$-module $\mathcal{V}$ is said to be* **pointed** *provided it is irreducible and it has at least one 1-dimensional weight space.*

One easily sees that a pointed torsion free module has only 1-dimensional weight spaces.

In [**6**], Lemire gave a construction of a family of pointed torsion free modules for algebras $\overset{0}{g}$ of type A. Fernando [**3**] found a single construction which could be modified to give a family of pointed torsion free modules for algebras $\overset{0}{g}$ of either type A or C. It begins with the Weyl algebra $\mathcal{W}_n$ of order n which can be realized as the associative subalgebra of $End_{\mathbf{C}}\mathbf{C}[x_1, \ldots, x_n]$ generated by $\{x_i, \partial_i | i = 1, \ldots, n\}$ where x_i and ∂_i are viewed as left multiplication by x_i and partial differentiation with respect to x_i, respectively. After we describe C_n and A_{n-1}, we embed them into $\mathcal{W}_n$.

Let $\mathbf{R}$ denote the set of real numbers, $\mathbf{R}^n$ real n-space, and $\{\epsilon_i | i = 1, \ldots, n\}$ the standard basis for $\mathbf{R}^n$. A root system for C_n is given by

$$\Phi(C_n) = \{\pm(\epsilon_i \pm \epsilon_j) | 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i | i = 1, \ldots, n\}$$

and one for A_{n-1} is contained in this as given by

$$\Phi(A_{n-1}) = \{\pm(\epsilon_i - \epsilon_j) | 1 \le i < j \le n\}.$$

A base for each of these root systems is given by $\Delta(C_n) = \{\alpha_1, \ldots, \alpha_n\} = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$ and $\Delta(A_{n-1}) = \{\alpha_1, \ldots, \alpha_{n-1}\} = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n\}$, respectively. We fix a basis for C_n of the form

$$\{x_{\pm(\epsilon_i \pm \epsilon_j)} | 1 \le i < j \le n\} \cup \{x_{\pm 2\epsilon_i} | i = 1, \ldots, n\} \cup \{h_{\alpha_i} | \alpha_i \in \Delta(C_n)\}$$

having the property that the map $\phi : C_n \longrightarrow \mathcal{W}_n$ given by

$$\phi(x_{\epsilon_i - \epsilon_j}) = x_i \partial_j \qquad \text{for } 1 \le i \ne j \le n, \tag{1.3}$$

$$\phi(x_{\epsilon_i + \epsilon_j}) = x_i x_j \qquad \text{for } i, j = 1, \ldots, n \tag{1.4}$$

$$\phi(x_{-(\epsilon_i + \epsilon_j)}) = \partial_i \partial_j \qquad \text{for } i, j = 1, \ldots, n \tag{1.5}$$

$$\phi(h_{\epsilon_i - \epsilon_{i+1}}) = x_i \partial_i - x_{i+1}\partial_{i+1} \qquad \text{for } i = 1, \ldots, n-1 \tag{1.6}$$

$$\phi(h_{2\epsilon_n}) = \frac{x_n \partial_n + \partial_n x_n}{2} \tag{1.7}$$

is a Lie algebra isomorphism. This allows us to consider $\mathbf{C}[x_1, \ldots, x_n]$ as a C_n module and to modify it somewhat to get the module $\overset{0}{M}(\vec{a})$ equal to the complex linear span

$$\overset{0}{M}(\vec{a}) = \text{lin. span}\{x^{\vec{b}} = x_1^{b_1} \cdots x_n^{b_n} | b_i - a_i \in \mathbf{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n}(b_i - a_i) \in 2\mathbf{Z}\} \tag{1.8}$$

where $\vec{a} = (a_1, \ldots, a_n)$ is fixed element of $\mathbf{C}^n$ with each $a_i \notin \mathbf{Z}$. It is easy to check that $\overset{0}{M}(\vec{a})$ is a pointed torsion free module for C_n.

Since the basis above contains

$$\{x_{\pm(\epsilon_i - \epsilon_j)} | 1 \leq i < j \leq n\} \cup \{h_{\alpha_i} | \alpha_i \in \Delta(A_{n-1})\}$$

a basis for A_{n-1}, the module $\overset{0}{M}(\vec{a})$ can be viewed as a module for A_{n-1}. This module has a pointed torsion free submodule $\overset{0}{N}(\vec{a})$ defined by the complex linear span

$$(1.9) \qquad \overset{0}{N}(\vec{a}) = \text{lin. span}\{x^{\vec{b}} = x_1^{b_1} \cdots x_n^{b_n} | b_i - a_i \in \mathbf{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n}(b_i - a_i) = 0\}.$$

In [3], Fernando proved that the only algebras $\overset{0}{g}$ which admit pointed torsion free modules are of type A and C. Based on this Britten and Lemire [1] classified all pointed modules for the algebras $\overset{0}{g}$. A special case of which is given here.

THEOREM 1.10 [1]. *The only pointed torsion free modules for the algebras $\overset{0}{g}$ are, up to equivalence, the C_n-modules $\overset{0}{M}(\vec{a})$ and the A_{n-1}-modules $\overset{0}{N}(\vec{a})$.*

Of central importance in our discussion are the pointed torsion free modules of A_1 which we now illustrated further.

EXAMPLE 1.11. *Let A_1 be the Lie algebra of 2×2 traceless matrices over $\mathbf{C}$ with basis*

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Fix two pointed torsion free modules $\overset{0}{N}(\vec{a})$ and $\overset{0}{N}(\vec{c})$ given by

$$\overset{0}{N}(\vec{a}) = \text{lin. span}\{x^{\vec{b}} = x_1^{b_1} x_2^{b_2} | b_i - a_i \in \mathbf{Z} \text{ for } i = 1,2 \text{ and } (b_1 - a_1) + (b_2 - a_2) = 0\}, \text{ and}$$
$$\overset{0}{N}(\vec{c}) = \text{lin. span}\{x^{\vec{d}} = x_1^{d_1} x_2^{d_2} | d_i - c_i \in \mathbf{Z} \text{ for } i = 1,2 \text{ and } (d_1 - c_1) + (d_2 - c_2) = 0\}.$$

Then in each case the operators are given by $\phi(x) = x_1\partial_2$, $\phi(y) = x_2\partial_1$, and $\phi(h) = x_1\partial_1 - x_2\partial_2$. We assume that $\overset{0}{N}(\vec{a})$ and $\overset{0}{N}(\vec{c})$ are related in such a way that $\lambda = a_1 - a_2 = c_1 - c_2$ is the weight of both $x^{\vec{a}}$ and $x^{\vec{c}}$ and $\mathcal{S} = s(s + \lambda + 1) = t(t + \lambda + 1) = \mathcal{T}$ where , $s = a_2$ and $t = c_2$. Note that

$$\phi(x)x^{\vec{b}} = (s - \ell)x_1^{b_1+1}x_2^{b_2-1}, \quad \phi(y)x^{\vec{b}} = (s + \lambda + \ell)x_1^{b_1-1}x_2^{b_2+1} \text{ and}$$
$$\phi(x)x^{\vec{d}} = (t - \ell)x_1^{d_1+1}x_2^{d_2-1}, \quad \phi(y)x^{\vec{d}} = (t + \lambda + \ell)x_1^{d_1-1}x_2^{d_2+1}$$

where $\ell = (b_1 - a_1) = (d_1 - c_1) \in \mathbf{Z}$. *Our assumption of* $\mathcal{S} = \mathcal{T}$ *means that the operator* $\phi(y)\phi(x)$ *acts in an equivalent manner on* $x^{\vec{a}}$ *and* $x^{\vec{c}}$ *as*

$$\phi(y)\phi(x)x^{\vec{a}} = \mathcal{S}x^{\vec{a}}, \text{ and}$$
$$\phi(y)\phi(x)x^{\vec{c}} = \mathcal{T}x^{\vec{c}}.$$

In fact, our assumptions $\lambda = a_1 - a_2 = c_1 - c_2$ *and* $\mathcal{S} = \mathcal{T}$ *imply that the modules* $\overset{0}{N}(\vec{a})$ *and* $\overset{0}{N}(\vec{c})$ *are equivalent. This equivalence is estalished by the map* $\eta : \overset{0}{N}(\vec{c}) \longrightarrow \overset{0}{N}(\vec{a})$ *defined by linearity and* $\eta(x_1^{c_1+\ell}x_2^{c_2-\ell}) = v_\ell$ *where*

$$v_\ell = \begin{cases} \dfrac{s\cdots(s-(\ell-1))}{t\cdots(t-(\ell-1))}x_1^{a_1+\ell}x_2^{a_2-\ell}, & \text{for } \ell > 0 \\ x_1^{a_1}x_2^{a_2}, & \\ \dfrac{(t+1)\cdots(t-\ell)}{(s+1)\cdots(s-\ell)}x_1^{a_1+\ell}x_2^{a_2-\ell}, & \text{for } \ell < 0 \end{cases}$$

since $\eta(zx_1^{c_1+\ell}x_2^{c_2-\ell}) = z\eta(x_1^{c_1+\ell}x_2^{c_2-\ell})$ *for* $z \in \{x, y, h\}$.

Conversely, it is easy to check that if the modules $\overset{0}{N}(\vec{a})$ *and* $\overset{0}{N}(\vec{c})$ *are equivalent the values* $\mathcal{S}$ *and* $\mathcal{T}$ *determined by the action of* yx *on the weight space* λ *must be equal. Thus the action of* yx *on a given weight space is independent of the choice of basis vector for that weight space.*

In the case of pointed torsion free C_n modules equivalence is completely determined by a single weight.

COROLLARY 1.12. *Two pointed torsion free* C_n *modules having a common weight* λ *are equivalent.*

PROOF: By Theorem 1.10, we may assume the two modules have the form given by (1.8), say $\overset{0}{M}(\vec{a})$ and $\overset{0}{M}(\vec{c})$ with $\lambda(h_{\alpha_i}) = \lambda_i = a_i - a_{i+1} = c_i - c_{i+1}$ for $i = 1, \ldots, n-1$ and $\lambda(h_{\alpha_n}) = \lambda_n = \frac{2a_n+1}{2} = \frac{2c_n+1}{2}$. $\overset{0}{M}(\vec{a})$ and $\overset{0}{M}(\vec{c})$ are equal because $a_i = c_i$ for $i = 1, \ldots, n$. ∎

To generalize the results of Example 1.11 to A_{n-1} modules, we present the following lemma.

LEMMA 1.13. *For* $n > 2$, *let* $\mathcal{V}$ *and* $\mathcal{U}$ *be two pointed torsion free* A_{n-1} *modules having a common weight* λ *with* $\lambda(h_{\alpha_1}) \neq 0$. *Let the corresponding weight spaces be* $\mathcal{V}_\lambda$ *and* $\mathcal{U}_\lambda$ *with* $v_\lambda \in \mathcal{V}_\lambda$ *and* $u_\lambda \in \mathcal{U}_\lambda$ *respective weight vectors. Then* $\mathcal{V}$ *and* $\mathcal{U}$ *are equivalent provided*

$$x_{-\alpha_2}x_{\alpha_2}v_\lambda = g_1v_\lambda,\ x_{-\alpha_2}x_{\alpha_2}u_\lambda = f_1v_\lambda \Longrightarrow g_1 = f_1, \text{ and} \tag{1.14}$$

$$x_{-(\alpha_1+\alpha_2)}x_{\alpha_+\alpha_2}v_\lambda = g_2v_\lambda,\ x_{-(\alpha_1+\alpha_2)}x_{\alpha_1+\alpha_2}u_\lambda = f_2u_\lambda \Longrightarrow g_2 = f_2. \tag{1.15}$$

PROOF: According to Theorem 1.10, $\mathcal{V}$ and $\mathcal{U}$ are equivalent to $\overset{0}{N}(\vec{a})$ and $\overset{0}{N}(\vec{c})$, respectively. We assume that $\mathcal{V} = \overset{0}{N}(\vec{a})$ and $\mathcal{U} = \overset{0}{N}(\vec{c})$ and show that our assumptions force $\overset{0}{N}(\vec{a}) = \overset{0}{N}(\vec{c})$. If $a_i = c_i$ for any $i = 1, \ldots, n$ then $\vec{a} = \vec{c}$ since $a_j - a_{j+1} = \lambda(h_{\alpha_j}) = c_j - c_{j+1}$ for all $j = 1, \ldots, n-1$. Observe that

$$a_3(a_3 + \lambda(h_{\alpha_2}) + 1) = g_1 = f_1 = c_3(c_3 + \lambda(h_{\alpha_1}) + 1), \text{ and}$$
$$a_3(a_3 + \lambda(h_{\alpha_1+\alpha_2}) + 1) = g_2 = f_2 = c_3(c_3 + \lambda(h_{\alpha_1+\alpha_2}) + 1)$$

imply $c_3 = a_3$, or $-a_3 - \lambda(h_{\alpha_2}) - 1$, and $c_3 = a_3$, or $-a_3 - \lambda(h_{\alpha_1+\alpha_2}) - 1$. Therefore, we can assume that $c_3 = -a_3 - \lambda(h_{\alpha_2}) - 1$ and $c_3 = -a_3 - \lambda(h_{\alpha_1+\alpha_2}) - 1$. From this it follows that $\lambda(h_{\alpha_1}) = 0$, contrary to assumption. ▮

Our aim in this article is to extend Theorem 1.10 and classify the pointed torsion free representations for non-twisted affine algebras. In Section 2, we construct $M(\mu, \vec{a})$ and $N(\mu, \vec{a})$ the analogs of $\overset{0}{M}(\vec{a})$ and $\overset{0}{N}(\vec{a})$ for the algebras $C_n^{(1)} = \hat{L}(C_n)$ and $A_{n-1}^{(1)} = \hat{L}(A_{n-1})$, respectively and state our Main Theorem. In Section 3, we prove this result for $C_n^{(1)}$. In Section 4, we restrict our attention to pointed torsion free modules for $A_1^{(1)}$ and produce some needed identities. Section 5 completes the proof of the Main Theorem for $A_1^{(1)}$, and Section 6 completes the proof by treating the case of $A_{n-1}^{(1)}$

2 Construction and Statement of Main Theorem. In this Section, we describe the modules $M(\mu, \vec{a})$ and $N(\mu, \vec{a})$ which completely determine the isomorphism classes of the pointed torsion free modules for non-twisted affine Lie algebras. We begin by making a few general remarks.

Let $\overset{0}{\mathcal{V}}$ be a module for $\overset{0}{g}$. It is easy to check that the tensor product $\mathcal{V} = L \otimes \overset{0}{\mathcal{V}}$ as $\mathbf{C}$ vector spaces is a module for $\hat{L}(\overset{0}{g})$ under the action given by

$$(t^{k'} \otimes x + \nu' c + \mu' d)(t^k \otimes v) = t^{k'+k} \otimes (xv) + \mu'(\mu + k)t^k \otimes v$$

where $\mu \in \mathbf{C}$ is some fixed complex number. There are a number of properties that $\mathcal{V}$ inherits from $\overset{0}{\mathcal{V}}$.

PROPOSITION 2.1. *Let $\mathcal{V}$ be the module constructed from $\overset{0}{\mathcal{V}}$ as above.*

(1) *If $\overset{0}{\mathcal{V}}$ is $\overset{0}{H}$-diagonalizable, then $\mathcal{V}$ is H-diagonalizable.*

(2) *If $\overset{0}{\mathcal{V}}$ is simple, (nontrivial but not necessarily H-diagonalizable) then $\mathcal{V}$ is simple.*

(3) *If $\overset{0}{\mathcal{V}}$ is $\overset{0}{H}$-diagonalizable with all it weight spaces 1-dimensional then all the weight spaces of $\mathcal{V}$ are 1-dimensional.*

PROOF: Items (1) and (3) are easily obtained consequences of the construction and although (2) is not much more we give a brief argument for it. Assume that $\overset{0}{\mathcal{V}}$ is nontrivial.

Let $v \in \overset{0}{\mathcal{V}}$ and $0 \neq u = \sum_{i=1}^{n} t^{k_i} \otimes v_i \in \mathcal{V}$. Let $U(\hat{L}(\overset{0}{g}))$ be the universal enveloping algebra of $\hat{L}(\overset{0}{g})$. It suffices to prove that $t^k \otimes v \in U(\hat{L}(\overset{0}{g}))u$. Let $w = \sum_{j=1}^{m} t^{k'_j} \otimes v'_j \in U(\hat{L}(\overset{0}{g}))$ where $k'_j \neq k'_i$ for $j \neq i$ with m being minimal. We claim that m must be 1 because $dw = \sum_{j=1}^{m}(k'_j + \mu)t^{k'_j} \otimes v'_j$ is nonzero if $m \neq 1$ and then $(k'_j + \mu)w - dw$ is an element in $U(\hat{L}(\overset{0}{g}))u$ involving a shorter summation contrary to the minimality of m. This means that $w = t^p \otimes v'$ for some integer p and some $v' \in \overset{0}{\mathcal{V}}$. Since $\overset{0}{\mathcal{V}}$ is simple there is some $x \in \overset{0}{g}$ such that $xv' = v$. Now, $(t^{k-p} \otimes x)w = t^k \otimes v$. ∎

This tensor product construction can be applied to $\overset{0}{M}(\vec{a})$ and $\overset{0}{N}(\vec{a})$. This amounts to extending the map $\phi : C_n \longrightarrow \mathcal{W}_n$ to $\hat{L}(\overset{0}{g})$ so that $\phi(c) = 0$, $\phi(d) = t\frac{\partial}{\partial t}$ and $\phi(t^k \otimes x) = t^k\phi(x)$ for all $x \in C_n$ and taking $M(\mu, \vec{a})$ and $N(\mu, \vec{a})$ to be given by

$$M(\mu,\vec{a}) = \{t^{\mu+k}x_1^{b_1}\cdots x_n^{b_n} \mid k, b_i - a_i \in \mathbf{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n}(b_i - a_i) \in 2\mathbf{Z}\}$$

$$N(\mu,\vec{a}) = \{t^{\mu+k}x_1^{b_1}\cdots x_n^{b_n} \mid k, b_i - a_i \in \mathbf{Z} \text{ for all } i \text{ and } \sum_{i=1}^{n}(b_i - a_i) = 0\}$$

where $\mu \in \mathbf{C}$ and $\vec{a} = (a_1, \ldots, a_n)$ is fixed element of $\mathbf{C}^n$ with each $a_i \notin \mathbf{Z}$.

The extended map ϕ seems fairly natural except perhaps for $\phi(c) = 0$ which seems more restrictive than required. The next lemma justifies this definition.

LEMMA 2.2. *Let $\mathcal{V}$ be a pointed torsion free module for $\hat{L}(\overset{0}{g})$. Then the central element c annihilates $\mathcal{V}$.*

PROOF: Let A_1 is a fixed subalgebra of $\overset{0}{g}$ with basis $\{x, y, h\}$ having the same multiplicative structure as in Example 1.11 and fix the non-degenerated symmetric bilinear form $(.|.)$ so that $(h|h) = 2$. Then $\hat{L}(A_1)$ is a subalgebra of $\hat{L}(\overset{0}{g})$ and $[t^\ell \otimes h, t^{-\ell} \otimes h] = 2\ell c$. Assume that $c\mathcal{V} \neq 0$. Since c is in the center of $\hat{L}(\overset{0}{g})$ we know that c acts as scalar multiplication by $K \neq 0$. Let v_0 be an arbitrary weight vector of $\mathcal{V}$ and let $a = (t^{-3} \otimes h)(t^2 \otimes h)(t \otimes h)$ and $b = (t^{-1} \otimes h)(t \otimes h)$. Since av_0 and bv_0 have the same weight as v_0 and weight spaces of $\mathcal{V}$ are 1-dimensional, we have

$$0 = [a, b]v_0 = 2cav_0 = 2Kav_0$$

However, v_0 is an arbitrary weight vector and so this tells us that $a\mathcal{V} = 0$.It now follows that

$$\{0\} = [(t^{-1} \otimes h), a]\mathcal{V} = -2(t^{-3} \otimes h)(t^2 \otimes h)c\mathcal{V} = -2K(t^{-3} \otimes h)(t^2 \otimes h)\mathcal{V}$$

and hence we have that $(t^{-3} \otimes h)(t^2 \otimes h)\mathcal{V} = \{0\}$. Using this we have that

$$\{0\} = [(t^{-2} \otimes h), (t^{-3} \otimes h)(t^2 \otimes h)]\mathcal{V} = -4(t^{-3} \otimes h)c\mathcal{V} = -4K(t^{-3} \otimes h)\mathcal{V}$$

and hence $(t^{-3} \otimes h)\mathcal{V} = \{0\}$. Finally, we have

$$\{0\} = [(t^3 \otimes h), (t^{-3} \otimes h)] = -6c\mathcal{V} = -6K\mathcal{V}.$$

This contradiction establishes that c must act trivially on $\mathcal{V}$. ∎

We now state our main result.

MAIN THEOREM. *The only non-twisted affine algebras $\hat{L}(\overset{0}{g})$ having pointed torsion free modules are those for which $\overset{0}{g}$ is either C_n or A_{n-1}. Moreover, the only such modules are $M(\mu,\vec{a})$ and $N(\mu,\vec{a})$ for $\hat{L}(C_n)$ and $\hat{L}(A_{n-1})$, respectively.*

The first statement is an immediate consequence of the analogous result from finite dimensional theory. The remainder of this paper is devoted to proving the second statement.

3 Proof of Main Theorem - the C_n case. In this Section, we show that the $\hat{L}(C_n)$ modules $M(\mu,\vec{a})$ constructed in Section 2 exhaust all pointed torsion free $\hat{L}(C_n)$ modules up to equivalence.

Let $\mathcal{V}$ be a pointed torsion free $\hat{L}(C_n)$ module and let Λ be a weight of $\mathcal{V}$. If $\{\alpha_0,\alpha_1,\ldots,\alpha_n\}$ denotes the simple roots of $\hat{L}(C_n)$ then $\mathcal{V}$ admits a weight space decomposition

$$\mathcal{V}=\sum_{k_i\in\mathbf{Z}}\mathcal{V}_{\Lambda+k_0\alpha_0+\cdots+k_n\alpha_n} \tag{3.1}$$

where each weight space is 1-dimensional. Setting $\mu=\Lambda(d)$ and $a_i=\Lambda(h_{\alpha_i}+\cdots+h_{\alpha_n})-\frac{1}{2}$ for $i=1,\ldots,n$ then the element $t^{\mu}x_1^{a_1}\cdots x_n^{a_n}$ is in the Λ weight space of the $\hat{L}(C_n)$ module $M(\mu,\vec{a})$. Our goal is to establish that $M(\mu,\vec{a})$ is equivalent to $\mathcal{V}$ as $\hat{L}(C_n)$ modules.

We first observe that $1\otimes C_n$ is a subalgebra of $\hat{L}(C_n)$ isomorphic to C_n. For any fixed integer k the space

$$\mathcal{V}_k=\sum_{k_1,\ldots,k_n\in\mathbf{Z}}\mathcal{V}_{\Lambda+k\alpha_0+k_1\alpha_1+\cdots+k_n\alpha_n}$$

is a pointed torsion free C_n-module. Also,

$$M(\mu,\vec{a})_k=\text{lin. span}\{t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}\mid \ell_i\in\mathbf{Z}\text{ and }\sum_{i=1}^{n}\ell_i\in 2\mathbf{Z}\}$$

is a pointed torsion free C_n with the same weight lattice as $\mathcal{V}_k$ and hence by Corollary 1.12 they are equivalent. We may therefore select a basis of vectors $v(k,k_1,\ldots,k_n)\in\mathcal{V}_{\Lambda+k\alpha_0+k_1\alpha_1+\cdots+k_n\alpha_n}$ such that the map

$$\psi_k: M(\mu,\vec{a})_k\longrightarrow\mathcal{V}_k$$

given by $\psi_k(t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n})=v(k,k_1,\ldots,k_n)$ where $k_i=2k+\ell_1+\cdots+\ell_i$ for $i=1,2,\ldots,n-1$ and $k_n=\frac{1}{2}(2k+\ell_1+\cdots+\ell_n)$ is a C_n module isomorphism. Clearly, the basis vectors $v(k,k_1,\ldots,k_n)$ are only determined up to a global constant - i.e. replacing each $v(k,k_1,\ldots,k_n)$ by $P_kv(k,k_1,\ldots,k_n)$ for some fixed nonzero complex number P_k depending on k would also yield an isomorphism.

Let $\theta=2\alpha_1+\cdots+2\alpha_{n-1}+\alpha_n$ denote the highest root of the finite root system $\{\alpha_1,\ldots,\alpha_n\}$. Then, following the construction of $\hat{L}(C_n)$, we have that $t^{-1}\otimes x_\theta$ and $t\otimes x_{-\theta}$ belong to the $-\alpha_0$ and α_0 root spaces of $\hat{L}(C_n)$, respectively. It follows then that $(t^{-1}\otimes x_\theta)v(k,0,\ldots,0)\in\mathcal{V}_{\Lambda+(k-1)\alpha_0}$. By adusting the global constants P_k, if need be, we may assume that $(t^{-1}\otimes x_\theta)v(k,0,\ldots,0)=v(k-1,0,\ldots,0)$. With this assumption we have completely determined a basis of $\mathcal{V}$ up to an initial choice of a nonzero vector $v(0,\ldots,0) \in \mathcal{V}_\Lambda$. We claim that the linear map

$$\psi : M(\mu,\vec{a}) \longrightarrow \mathcal{V}$$

extending each ψ_k is an $\hat{L}(C_n)$ module isomorphism.

Since $\hat{L}(C_n)$ is generated by the elements of

$$\{1\otimes x_{\pm\alpha_i} \mid i=1,\ldots,n\} \cup \{t^{-1}\otimes x_\theta, t\otimes x_{-\theta}\} \cup \{d\},$$

it suffices to show that the operators corresponding to these elements commute with ψ. Since ψ is an extension of each ψ_k we know that $(1\otimes x_{\pm\alpha_i})\circ\psi = \psi\circ(1\otimes x_{\pm\alpha_i})$. Also, since ψ preserves weight spaces we have $d\circ\psi = \psi\circ d$. It remains to be shown that $(t^{-1}\otimes x_\theta)\circ\psi = \psi\circ(t^{-1}\otimes x_\theta)$ and $(t\otimes x_{-\theta})\circ\psi = \psi\circ(t\otimes x_{-\theta})$.

Let $\overset{0}{C}$ denote the subalgebra of $\hat{L}(C_n)$ generated by

$$\{1\otimes x_{\pm\alpha_i} \mid i=1,\ldots,n-1\} \cup \{t^{-1}\otimes x_\theta, t\otimes x_{-\theta}\}.$$

The algebra $(\overset{0}{C} + \mathbf{C}c)/\mathbf{C}c$ is isomorphic to C_n. Since the central element c acts trivially on any pointed torsion free $\hat{L}(C_n)$ module, the subspace

$$\mathcal{V}^{k_n} = \sum_{k_0,\ldots,k_{n-1}\in\mathbf{Z}} \mathcal{V}_{\Lambda+k_0\alpha_0+k_1\alpha_1+\cdots+k_n\alpha_n}$$

can be considered to be a pointed torsion free C_n module. Similarly, the subspace

$$M(\mu,\vec{a})^{k_n} = \text{lin. span}\{t^{\mu+\ell_0}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n} \mid \ell_i\in\mathbf{Z},\ \sum_{i=1}^n \ell_i \in 2\mathbf{Z} \text{ and } 2\ell_0+\ell_1+\cdots+\ell_n = 2k_n\}$$

is a pointed torsion free C_n module having the weight $\Lambda + k_n\alpha_n$ in common with $\mathcal{V}^{k_n}$ and hence they are equivalent as C_n modules.

LEMMA 3.2. *If $v(k,k_1,\ldots,k_n) = \psi(t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n})$ then*

$$(t^{-1}\otimes x_\theta)v(k,k_1,\ldots,k_n) = \psi((t^{-1}\otimes x_\theta)(t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}))$$

for all k if and only if

$$(t\otimes x_{-\theta})v(k,k_1,\ldots,k_n) = \psi((t\otimes x_{-\theta})(t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}))$$

for all k.

PROOF: By assumption $v(k,k_1,\ldots,k_n)$ belongs to the $\Lambda + k\alpha_0 + \cdots + k_n\alpha_n$ weight space of $\mathcal{V}^{k_n}$ and $t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}$ belongs to the $\Lambda + k\alpha_0 + \cdots + k_n\alpha_n$ weight space of $M(\mu,\vec{a})^{k_n}$ where $k_i = 2k+\ell_1+\cdots+\ell_i$ for $i=1,2,\ldots,n-1$ and $k_n = \frac{1}{2}(2k+\ell_1+\cdots+\ell_n)$.

Since $\mathcal{V}^{k_n}$ and $M(\mu,\vec{a})^{k_n}$ are equivalent and $v(k,k_1,\ldots,k_n)$ and $t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}$ are in corresponding 1-dimensional weight spaces, we know

$$(t^{-1}\otimes x_\theta)(t\otimes x_{-\theta})v(k,k_1,\ldots,k_n)=\psi((t^{-1}\otimes x_\theta)(t\otimes x_{-\theta})(t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}))$$

for all k. Since $t^{-1}\otimes x_\theta$ belongs to the $-\alpha_0$ root space of $\hat{L}(C_n)$, we have $(t^{-1}\otimes x_\theta)v(k,k_1,\ldots,k_n)$ is a complex multiple of $v(k-1,k_1,\ldots,k_n)$ and hence if

$$\begin{aligned}(t^{-1}\otimes x_\theta)v(k,k_1,\ldots,k_n)=&(t^{-1}\otimes x_\theta)\psi((t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}))\\ =&\psi((t^{-1}\otimes x_\theta)(t^{\mu+k}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}))\end{aligned}$$

then

$$\begin{aligned}(t\otimes x_{-\theta})v(k-1,k_1,\ldots,k_n)&=(t\otimes x_{-\theta})\psi((t^{\mu+k-1}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n}))\\ &=\psi((t\otimes x_{-\theta})(t^{\mu+k-1}x_1^{a_1+\ell_1}\cdots x_n^{a_n+\ell_n})).\end{aligned}$$

By interchanging the roles of $t^{-1}\otimes x_\theta$ and $t\otimes x_{-\theta}$ as well as $v(k,k_1,\ldots,k_n)$ and $v(k-1,k_1,\ldots,k_n)$, one obtains the converse. ∎

THEOREM 3.3. *The linear map*

$$\psi: M(\mu,\vec{a})\longrightarrow\mathcal{V}$$

extending each ψ_k is an $\hat{L}(C_n)$ module isomorphism.

PROOF: All that remains to be shown is that for each n-tuple $(k_1,\ldots,k_n)\in\mathbf{Z}^n$ one of the two equalities of Lemma 3.2 hold for all k. By our choice of the basis of $\mathcal{V}$, we have that

$$(t^{-1}\otimes x_\theta)v(k,0,\ldots,0)=v(k-1,0,\ldots,0)$$

for all $k\in\mathbf{Z}$. Also, since $\theta=2\epsilon_1$ and by (1.4) $\phi(x_{2\epsilon_1})=x_1^2$ we have

$$(t^{-1}\otimes x_\theta)t^{\mu+k}x_1^{a_1-2k}x_2^{a_2}\cdots x_n^{a_n}=t^{\mu+k-1}x_1^{a_1-2(k-1)}x_2^{a_2}\cdots x_n^{a_n}$$

for all $k\in\mathbf{Z}$. Using the definition of the map ψ,we can conclude that for all $k\in\mathbf{Z}$

$$\begin{aligned}(t^{-1}\otimes x_\theta)v(k,0,\ldots,0)&=(t^{-1}\otimes x_\theta)\psi(t^{\mu+k}x_1^{a_1-2k}x_2^{a_2}\cdots x_n^{a_n})\\ &=\psi((t^{-1}\otimes x_\theta)t^{\mu+k}x_1^{a_1-2k}x_2^{a_2}\cdots x_n^{a_n}).\end{aligned}$$

By Lemma 3.2, this implies that for all $k\in\mathbf{Z}$

$$\begin{aligned}(t\otimes x_{-\theta})v(k,0,\ldots,0)&=(t\otimes x_{-\theta})\psi(t^{\mu+k}x_1^{a_1-2k}x_2^{a_2}\cdots x_n^{a_n})\\ &=\psi((t\otimes x_{-\theta})t^{\mu+k}x_1^{a_1-2k}x_2^{a_2}\cdots x_n^{a_n}).\end{aligned}$$

We now use an inductive argument to prove the result for the weight $\Lambda+k\alpha_0+k_1\alpha_1+\cdots+k_n\alpha_n$ when $\sum_{i=1}^n|k_i|>0$. Consider first the case of $\sum_{i=1}^n|k_i|>0$ where $k_i>0$ for some $i = 1, \ldots, n$. Then since $\mathcal{V}_k$ is equivalent to $M(\mu, \vec{a})_k$ we have from $v(k, k_1, \ldots, k_n) = \psi_k(t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n})$ that

$$v(k, k_1, \ldots, k_n) = \frac{1 \otimes x_{\alpha_i}}{\kappa_i} v(k, k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n)$$

where

$$\kappa_i = \begin{cases} a_{i+1} + \ell_{i+1} & \text{for } i = 1, \ldots, n-1 \\ 1 & \text{for } i = n. \end{cases}$$

Using the inductive hypothesis and the fact that x_{α_i} commutes with x_θ, we have

$$\begin{aligned}
&(t^{-1} \otimes x_\theta) v(k, k_1, \ldots, k_n) \\
&\quad = (t^{-1} \otimes x_\theta) \frac{1 \otimes x_{\alpha_i}}{\kappa_i} v(k, k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_n) \\
&\quad = \frac{1 \otimes x_{\alpha_i}}{\kappa_i} (t^{-1} \otimes x_\theta) \psi_k(t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_i^{a_i+\ell_i+1} x_{i+1}^{a_{i+1}+\ell_{i+1}-1} \cdots x_n^{a_n+\ell_n}) \\
&\quad = \frac{1 \otimes x_{\alpha_i}}{\kappa_i} \psi_k((t^{-1} \otimes x_\theta) t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_i^{a_i+\ell_i+1} x_{i+1}^{a_{i+1}+\ell_{i+1}-1} \cdots x_n^{a_n+\ell_n}) \\
&\quad = \psi_k(\frac{1 \otimes x_{\alpha_i}}{\kappa_i} (t^{-1} \otimes x_\theta) t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_i^{a_i+\ell_i+1} x_{i+1}^{a_{i+1}+\ell_{i+1}-1} \cdots x_n^{a_n+\ell_n}) \\
&\quad = \psi_k((t^{-1} \otimes x_\theta) \frac{1 \otimes x_{\alpha_i}}{\kappa_i} t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_i^{a_i+\ell_i+1} x_{i+1}^{a_{i+1}+\ell_{i+1}-1} \cdots x_n^{a_n+\ell_n}) \\
&\quad = \psi_k((t^{-1} \otimes x_\theta) t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}).
\end{aligned}$$

Therefore, by Lemma 3.2, we conclude that for all $k \in \mathbf{Z}$.

$$\begin{aligned}
(t \otimes x_{-\theta}) v(k, k_1, \ldots, k_n) &= (t \otimes x_{-\theta}) \psi(t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}) \\
&= \psi((t \otimes x_{-\theta}) t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}).
\end{aligned}$$

The argument in the case of $k_i < 0$ to establish

$$\begin{aligned}
(t \otimes x_{-\theta}) v(k, k_1, \ldots, k_n) &= (t \otimes x_{-\theta}) \psi(t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}) \\
&= \psi((t \otimes x_{-\theta}) t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}).
\end{aligned}$$

is analogus. Once we have this Lemma 3.2 gives us

$$\begin{aligned}
(t^{-1} \otimes x_\theta) v(k, k_1, \ldots, k_n) &= (t^{-1} \otimes x_\theta) \psi(t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}) \\
&= \psi((t^{-1} \otimes x_\theta) t^{\mu+k} x_1^{a_1+\ell_1} \cdots x_n^{a_n+\ell_n}).
\end{aligned}$$

Thus both $t^{-1} \otimes x_\theta$ and $t \otimes x_{-\theta}$ commute with ψ. Therefore, $\mathcal{V}$ is equivalent to $M(\mu, \vec{a})$ as claimed. ∎

4 Identities. Our aim in this section is to produce a set of identities which must hold in pointed torsion free $A_1^{(1)}$ modules $\mathcal{V}$. These identities will be used in Section 5 to prove the Main Theorem for the algebra $A_1^{(1)}$. We fix a basis for an underlying finite dimensional simple algebra A_1 as $\{x, y, h\}$ with multiplicative structure as in Example 1.11 and let $\{\alpha_0, \alpha_1\}$ denote the simple roots of $A_1^{(1)} = \hat{L}(A_1)$. We assume that $\mathcal{V}$ is a pointed torsion free $A_1^{(1)}$ module. This tells us that $\mathcal{V}$ has a weight space decomposition given by

$$\mathcal{V} = \sum_{k_0, k_1 \in \mathbf{Z}} \mathcal{V}_{\Lambda + k_0\alpha_0 + k_1\alpha_1}.$$

Let $\delta = \alpha_0 + \alpha_1$, $\Lambda(d) = \mu$ where d is as in (1.1) and let $\Lambda(h_{\alpha_1}) = \lambda$. Set $k = k_0$ and $\ell = k_1 - k_0$ so that $(\Lambda + k_0\delta + (k_1 - k_0)\alpha_1)(d) = \mu + k$ and $(\Lambda + k_0\delta + (k_1 - k_0)\alpha_1)(h_{\alpha_1}) = \lambda + 2\ell$. This allows us to denote $\Lambda + k_0\alpha + k_1\alpha_1$ by $(\mu, \lambda) + (k, 2\ell)$. Each weight space $\mathcal{V}_{(\mu,\lambda)+(k,2\ell)}$ is 1-dimensional and

$$\{u(k, \ell) \mid 0 \neq u(k, \ell) \in \mathcal{V}_{(\mu,\lambda)+(k,2\ell)} \text{ and } k, \ell \in \mathbf{Z}\}$$

is a basis of $\mathcal{V}$ and hence we can write

$$\mathcal{V} = \sum_{k,\ell \in \mathbf{Z}} \mathcal{V}_{(\mu,\lambda)+(k,2\ell)} = \sum_{k,\ell \in \mathbf{Z}} \mathbf{C}u(k, \ell)$$

We can express $\mathcal{V}$ as a direct sum of pointed torsion free $1 \otimes A_1$ modules

$$\mathcal{V}_k = \sum_{\ell \in \mathbf{Z}} \mathbf{C}u(k, \ell).$$

According to the remarks made in Example 1.11, we may assume that the basis vectors $u(k, \ell)$ have been selected so that for some $s_k \in \mathbf{C}$ we have

$$(4.1) \qquad (1 \otimes x)u(k, \ell) = (s_k - \ell)u(k, \ell + 1),$$

$$(4.2) \qquad (1 \otimes y)u(k, \ell) = (s_k + \lambda + \ell)u(k, \ell - 1),$$

$$(4.3) \qquad (1 \otimes h)u(k, \ell) = (\lambda + 2l)u(k, \ell).$$

Also by making use of a single normalizing factor for each $\mathcal{V}_k$, we can assume further that

$$(4.4) \qquad (t^{-1} \otimes x)u(-\ell, \ell) = (s_{-\ell} - \ell)u(-\ell - 1, \ell + 1).$$

Moreover, we know there exist complex numbers $A(k, \ell), B(k, \ell), C(k, \ell)$ and $D(k, \ell)$ such that

$$(4.5) \qquad (t^{-1} \otimes x)u(k, \ell) = A(k, \ell)u(k - 1, \ell + 1),$$

$$(4.6) \qquad (t \otimes y)u(k, \ell) = B(k, \ell)u(k + 1, \ell - 1),$$

$$(4.7) \qquad (t \otimes h)u(k, \ell) = C(k, \ell)u(k + 1, \ell),$$

$$(4.8) \qquad (t^{-1} \otimes h)u(k, \ell) = D(k, \ell)u(k - 1, \ell).$$

Note according to (4.4) and (4.5)

$$(4.9) \qquad A(-\ell, \ell) = s_{-\ell} - \ell$$

for all l. We can obtain further relationships among these coefficients by making use of commutator products.

LEMMA 4.10. *Let $A(k,\ell), B(k,\ell), C(k,\ell)$ and $D(k,\ell)$ be defined by (4.5)-(4.8). Then*

$$A(k,\ell)(s_{k-1}-\ell-1)=(s_k-\ell)A(k,\ell+1), \tag{4.11}$$

$$B(k,\ell)(s_{k+1}+\lambda+\ell-1)=(s_k+\lambda+\ell)B(k,\ell-1), \tag{4.12}$$

$$C(k,\ell)D(k+1,\ell)=D(k,\ell)C(k-1,\ell), \tag{4.13}$$

$$2(s_k-\ell)=A(k,\ell)C(k-1,\ell+1)-C(k,\ell)A(k+1,\ell), \tag{4.14}$$

$$C(k,\ell)=B(k,\ell)(s_{k+1}-\ell+1)-(s_k-\ell)B(k,\ell+1), \text{ and} \tag{4.15}$$

$$D(k,\ell)=(s_k+\lambda+\ell)A(k,\ell-1)-A(k,\ell)(s_{k-1}+\lambda+\ell+1). \tag{4.16}$$

PROOF: The equations (4.11) through (4.16) follow, respectively, from the action of the commutators $[1\otimes x, t^{-1}\otimes x]$, $[1\otimes y, t\otimes y]$, $[t^{-1}\otimes h, t\otimes h]$, $[t\otimes h, t^{-1}\otimes x]$, $[1\otimes x, t\otimes y]$, and $[t^{-1}\otimes x, 1\otimes y]$ on the weight vector $u(k,\ell)$ as shown in the two cases

$$0=[t^{-1}\otimes h, t\otimes h]u(k,\ell)=(C(k,\ell)D(k+1,\ell)-D(k,\ell)C(k-1,\ell))u(k,\ell)$$

and

$$\begin{aligned} D(k,\ell)u(k-1,\ell)&=(t^{-1}\otimes h)u(k,\ell)=[t^{-1}\otimes x, 1\otimes y]u(k,\ell)\\ &=((s_k+\lambda+\ell)A(k,\ell-1)-A(k,\ell)(s_{k-1}+\lambda+\ell+1))u(k-1,\ell). \end{aligned}$$

The other cases are similar. ∎

We find it convenient to introduce the notation

$$\mathcal{S}_k=s_k(s_k+\lambda+1) \text{ and} \tag{4.17}$$

$$\sigma(k,\ell)=\mathcal{S}_k-\mathcal{S}_{k-1}+\lambda+2\ell. \tag{4.18}$$

One should note that the torsion free assumption on $\mathcal{V}$ implies that for all $\ell\in\mathbf{Z}$ we have $s_k+\lambda+\ell\neq 0$ and $s_k-\ell\neq 0$.

LEMMA 4.19. *Using the notation of (4.17) and (4.18) we have*

$$C(k,\ell)=\sigma(k+1,\ell)\frac{B(k,\ell)}{s_{k+1}+\lambda+\ell} \text{ and} \tag{4.20}$$

$$D(k+1,\ell)=\sigma(k+1,\ell)\frac{A(k+1,\ell)}{s_k-\ell}. \tag{4.21}$$

PROOF: From (4.12), it follows that $B(k,\ell+1)(s_{k+1}+\lambda+\ell)=B(k,\ell)(s_k+\lambda+\ell+1)$. Using this with (4.15) gives

$$C(k,\ell)=B(k,\ell)(s_{k+1}-\ell+1)\frac{(s_{k+1}+\lambda+\ell)}{(s_{k+1}+\lambda+\ell)}-(s_k-\ell)\frac{B(k,\ell)(s_k+\lambda+\ell+1)}{(s_{k+1}+\lambda+\ell)}$$

from which (4.20) easily follows. Similarly from (4.11), it follows that $A(k+1,\ell-1)(s_k-\ell)=(s_{k+1}-\ell+1)A(k+1,\ell)$ and using this in the expression for $D(k+1)$ obtained from (4.16) we find

$$D(k+1,\ell)=(s_{k+1}+\lambda+\ell)\frac{A(k+1,\ell)(s_{k+1}-\ell+1)}{(s_k-\ell)}-A(k+1,\ell)(s_k+\lambda+\ell+1)\frac{(s_k-\ell)}{(s_k-\ell)}$$

and combining terms (4.21) follows. ∎

LEMMA 4.22.

$$A(k,\ell)=\begin{cases}(s_k-\ell)\displaystyle\prod_{j=1}^{k+\ell}\frac{s_{k-1}+k-j}{s_k+k-j} & \text{for } k+\ell>0\\ (s_k-\ell) & \text{for } k+\ell=0\\ (s_k-\ell)\displaystyle\prod_{j=0}^{-k-\ell-1}\frac{s_k+k+j}{s_{k-1}+k+j} & \text{for } k+\ell<0.\end{cases}$$

PROOF: By choice of basis $A(-\ell,\ell)=s_{-\ell}-\ell$ and hence the result is true for $k+\ell=0$.

Assume first that $k+\ell>0$. From (4.11) we know $A(k,-k)(s_{k-1}+k-1)=(s_k+k)A(k,-k+1)$ which simplifies to $(s_{k-1}+k-1)=A(k,-k+1)$ so that beginning our inductive argument with $k+\ell=1$ we have

$$A(k,-k+1)=s_{k-1}+k-1=(s_k-\ell)\prod_{j=1}^{k+\ell}\frac{s_{k-1}+k-j}{s_k+k-j}.$$

Assume inductively that for $q>0$ we have $A(k,-k+q)=(s_k+k-q)\prod_{j=1}^{q}\frac{s_{k-1}+k-j}{s_k+k-j}$. Then setting $\ell=-k+q$ in (4.11), we get

$$\begin{aligned}A(k,-k+q+1)&=(s_{k-1}+k-q-1)\frac{A(k,-k+q)}{s_k+k-q}=(s_{k-1}+k-q-1)\prod_{j=1}^{q}\frac{s_{k-1}+k-j}{s_k+k-j}\\&=(s_k+k-q-1)\prod_{j=1}^{q+1}\frac{s_{k-1}+k-j}{s_k+k-j}.\end{aligned}$$

The $k+\ell>0$ case now follows.

The case of $k+\ell<0$ is proven in a similar manner. ∎

LEMMA 4.23.

$$D(k+1,\ell)=\begin{cases}\sigma(k+1,\ell)\displaystyle\prod_{j=1}^{k+\ell}\frac{s_k+k+1-j}{s_{k+1}+k+1-j} & \text{for } k+\ell>0\\ \sigma(k+1,\ell) & \text{for } k+\ell=0\\ \sigma(k+1,\ell)\displaystyle\prod_{j=0}^{-k-\ell-1}\frac{s_{k+1}+k+1-j}{s_k+k+1-j} & \text{for } k+\ell<0.\end{cases}$$

PROOF: This follows immediately from equation (4.21) and Lemma 4.22. ∎

LEMMA 4.24. *$C(k,\ell)D(k+1,\ell)=C(-\ell,\ell)D(-\ell+1,\ell)$ for all k and ℓ.*

PROOF: This is an immediate consequence of (4.13). ∎

LEMMA 4.25. *If $C = C(0,0)$ and either $k+\ell = 0$ or $\sigma(k+1,\ell) \neq 0$, then*

$$C(k,\ell) = \begin{cases} (C+2\ell)\dfrac{\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)} \displaystyle\prod_{j=1}^{k+\ell} \dfrac{s_{k+1}+k+1-j}{s_k+k+1-j} & \text{for } k+\ell > 0 \\ (C+2\ell) & \text{for } k+\ell = 0 \\ (C+2\ell)\dfrac{\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)} \displaystyle\prod_{j=0}^{-k-\ell-1} \dfrac{s_k+k+1+j}{s_{k+1}+k+1+j} & \text{for } k+\ell < 0. \end{cases}$$

PROOF: We prove this first when $k = -\ell$. In this case (4.11) gives $A(-\ell+1,\ell-1)(s_{-\ell}-\ell) = (s_{-\ell+1}-\ell+1)A(-\ell+1,\ell)$ which according to (4.9) implies $A(-\ell+1,\ell) = (s_{-\ell}-\ell)$. Using this and (4.9) in (4.14) gives $2 = C(-\ell-1,\ell+1) - C(-\ell,\ell)$ and hence $C(-\ell,\ell) = C(0,0)+2\ell$.

We now assume that $\sigma(k+1,\ell) \neq 0$ and hence from (4.21) $D(k+1,\ell) \neq 0$. From Lemma 4.24, we have that

$$C(k,\ell) = \frac{C(-\ell,\ell)D(-\ell+1,\ell)}{D(k+1,\ell)} = \frac{(C+2\ell)D(-\ell+1,\ell)}{D(k+1,\ell)}.$$

The result now follows from Lemma 4.23. ∎

LEMMA 4.26. *If $C = C(0,0)$ and $\sigma(k+1,\ell) \neq 0$, then*

$$B(k,\ell) = \begin{cases} (s_{k+1}+\lambda+\ell)\dfrac{(C+2\ell)\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)^2} \displaystyle\prod_{j=1}^{k+\ell} \dfrac{s_{k+1}+k+1-j}{s_k+k+1-j} & \text{for } k+\ell > 0 \\ (s_{k+1}+\lambda+\ell)\dfrac{(C+2\ell)}{\sigma(k+1,\ell)} & \text{for } k+\ell = 0 \\ (s_{k+1}+\lambda+\ell)\dfrac{(C+2\ell)\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)^2} \displaystyle\prod_{j=0}^{-k-\ell-1} \dfrac{s_k+k+1+j}{s_{k+1}+k+1+j} & \text{for } k+\ell < 0. \end{cases}$$

PROOF: This follows immediately from (4.20) and Lemma 4.25. ∎

LEMMA 4.27. *If $C = C(0,0)$ and $\sigma(k+1,\ell) \neq 0$, then*

$$2 = (C+2(\ell+1))\frac{\sigma(-\ell,\ell+1)}{\sigma(k,\ell+1)} - (C+2\ell)\frac{\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)} \tag{4.28}$$

and if in addition $\sigma(k+1,\ell+1) \neq 0$ then

$$\begin{aligned}(C+2\ell)\frac{\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)} &= (s_{k+1}+\lambda+\ell)(s_{k+1}-\ell+1)(C+2\ell)\frac{\sigma(-\ell+1,\ell)}{\sigma(k+1,\ell)^2} \\ &\quad - (s_{k+1}+\lambda+\ell+1)(s_{k+1}-\ell)(C+2(\ell+1))\frac{\sigma(-\ell,\ell+1)}{\sigma(k+1,\ell+1)^2}.\end{aligned} \tag{4.29}$$

PROOF: Equation (4.28) is proven by substituting the values of Lemma 4.22 and Lemma 4.25 into (4.14) under the various conditions $k+\ell > 0, k+\ell = 0, k+\ell = -1$ and $k+\ell < -1$. In the case of $k+\ell > 0$, we have

$$2(s_k - \ell) = (s_k - \ell) \prod_{j=1}^{k+\ell} \frac{s_{k-1} + k - j}{s_k + k - j} (C + 2(\ell+1)) \frac{\sigma(-\ell, \ell+1)}{\sigma(k, \ell+1)} \prod_{j=1}^{k+\ell} \frac{s_k + k - j}{s_{k-1} + k - j}$$
$$- (s_{k+1} - \ell) \prod_{j=1}^{k+\ell+1} \frac{s_k + k + 1 - j}{s_{k+1} + k + 1 - j} (C + 2\ell) \frac{\sigma(-\ell+1, \ell)}{\sigma(k+1, \ell)} \prod_{j=1}^{k+\ell} \frac{s_{k+1} + k + 1 - j}{s_k + k + 1 - j}.$$

and the result follows after simplification. Equation (4.29) is proved in a similar manner by substituting the values of Lemma 4.25 and Lemma 4.26 into (4.15) and simplifying. We omit the display of further detail. ∎

5 Proof of Main Theorem for $A_1^{(1)}$. In this Section, we continue our study of the module

$$\mathcal{V} = \sum_{k,\ell \in \mathbf{Z}} \mathcal{V}_{(\mu,\lambda)+(k,2\ell)} = \oplus \sum_{k \in \mathbf{Z}} \mathcal{V}_k$$

as described in Section 4. Our first aim is to show that the A_1 module $\mathcal{V}_k$ is equivalent to $\mathcal{V}_0$ for all k. This requires showing that

$$(5.1) \qquad \mathcal{S}_k = \mathcal{S}_0 \text{ for all } k.$$

As a step toward this end we show that $\mathcal{S}_k - \mathcal{S}_{k-1} = \mathcal{S}_1 - \mathcal{S}_0$ for all k. There is one case in which this is relatively easy to obtain.

LEMMA 5.2. *If $C(k_0, \ell_0) = 0$ for some k_0 and ℓ_0 then $\mathcal{S}_k - \mathcal{S}_{k-1} = \mathcal{S}_1 - \mathcal{S}_0$ for all k.*

PROOF: Recall that our torsion free assumption implies that $s_k - \ell, s_k + \lambda + \ell, A(k,\ell)$ and $B(k,\ell)$ are all nonzero for all k and ℓ. It follows from (4.20) and (4.21) that

$$(5.3) \qquad C(k_0, \ell_0) = 0 \iff \sigma(k_0+1, \ell_0) = 0 \iff D(k_0+1, \ell_0) = 0.$$

From this and (4.13), we can conclude that

$$(5.4) \qquad C(k, \ell_0) = \sigma(k, \ell_0) = D(k, \ell_0) = 0$$

for all k and hence $\mathcal{S}_k - \mathcal{S}_{k-1} = \mathcal{S}_1 - \mathcal{S}_0$. ∎

Now we assume that $C(k,\ell)$ is never zero and use equation (4.28) to produce six equations which we solve to obtain $\mathcal{S}_{k+1} - \mathcal{S}_k = \mathcal{S}_1 - \mathcal{S}_0$ for all k. We then use equation (4.29) to get

$$\mathcal{S}_{k+1} = \mathcal{S}_k$$

for all k. By Lemma 4.25, our assumption that $C(k,\ell)$ is never zero implies that C is not an even integer.

LEMMA 5.5. *Let*

$$X = \mathcal{S}_1 - \mathcal{S}_0 + \lambda,$$
$$T = \mathcal{S}_2 - \mathcal{S}_1 + \lambda,$$
$$U = \mathcal{S}_0 - \mathcal{S}_{-1} + \lambda, \text{ and}$$
$$V = \mathcal{S}_{-1} - \mathcal{S}_{-2} + \lambda$$

Then

$$2 = (C+2)\frac{U+2}{V+2} - C\frac{X}{U}, \tag{5.6}$$

$$2 = (C+2)\frac{U+2}{X+2} - C\frac{X}{T}, \tag{5.7}$$

$$2 = (C+4)\frac{V+4}{U+4} - (C+2)\frac{U+2}{X+2}, \tag{5.8}$$

$$2 = (C+4)\frac{V+4}{X+4} - (C+2)\frac{U+2}{T+2}, \tag{5.9}$$

$$2 = C\frac{X}{V} - (C-2)\frac{T-2}{U-2}, \text{ and} \tag{5.10}$$

$$2 = C\frac{X}{U} - (C-2)\frac{T-2}{X-2}. \tag{5.11}$$

PROOF: These equations follow immediately from (4.28) by making the substitutions

$$(k,\ell) = (-1,0),(1,0),(0,1),(1,1),(-1,-1),(0,-1)$$

and then using the definitions of X, T, U, V. ∎

THEOREM 5.12. *If C, X, T, U, and V form a solution to the system (5.6)-(5.11) and none of these are even integers, then $X = T = U = V$.*

PROOF: First we notice that if $U = V$ then (5.6) and (5.7) imply that $X = T = U = V$. The remainder of the proof consists of showing that equations (5.5)-(5.11) together with the assumption $U \neq V$ leads to a contradiction and hence the only consistent solution to this system of equations is $X = T = U = V$.

Equation (5.6) is equivalent to

$$2U(V-U) = C[U(U+2) - X(V+2)] \tag{5.13.}$$

If $U(U+2) - X(V+2) = 0$ then (5.13) implies that $U = 0$ or $U = V$ contrary to assumption. Therefore, C is given by

$$C = \frac{2U(V-U)}{[U(U+2) - X(V+2)]} \tag{5.14}$$

We substitute this value for C into (5.7) to obtain

$$\begin{aligned}2 &= \frac{[\frac{-2U(U-V)}{U(U+2)-X(V+2)}+2](U+2)}{X+2} - \frac{-2U(U-V)}{U(U+2)-X(V+2)}\frac{X}{T} \\ &= \frac{T(U+2)[-2U(U-V)+2U(U+2)-2X(V+2)]}{T(X+2)[U(U+2)-X(V+2)]} \\ &\quad + \frac{2U(U-V)X(X+2)}{T(X+2)(U(U+2)-X(V+2))}.\end{aligned}$$

from which it follows that

$$\begin{aligned}T[(X+2)(U(U+2)-X(V+2))+(U+2)(U(U-V)-U(U+2)+X(V+2))] \\ = U(U-V)X(X+2)\end{aligned}$$

and since the right hand side of this equation is nonzero, the coefficient of T is also nonzero. Therefore, T is given by

(5.15)
$$\begin{aligned}T &= \frac{U(U-V)X(X+2)}{(X+2)(U(U+2)-X(V+2))+(U+2)(U(U-V)-U(U+2)+X(V+2))} \\ &= \frac{-(-X^2U^2+UVX^2-2U^2X+2UVX)}{U^2X+4UX-VX^2-2X^2-U^2V-2UV+UVX} \\ &= \frac{X^2(U^2-UV)+X(2U^2-2UV)}{X^2(-V-2)+X(U^2+4U+UV)+(-U^2V-2UV)}\end{aligned}$$

If we multiply equation (5.8) by $(U+4)(X+2)$ and rearrange terms we get

$$C+2 = \frac{2(X+2)(U-V)}{(V+4)(X+2)-(U+2)(U+4)} \tag{5.16}$$

If we substitute the value for C given by (5.14) into (5.16) and simplify we get

$$\frac{-2U(U-V)}{U^2+2U-VX-2X}+2 = \frac{2(X+2)(U-V)}{(V+4)(X+2)-(U+2)(U+4)}.$$

From this we can obtain a polynomial which is quadratic in X and is equal to 0 written as follows

$$\begin{aligned}X^2(2UV+4U-4V^2-16V-16)+X(-16V+4U^2V+2UV^2+32UV+48U-2U^3-8V^2) \\ +(16UV-32U^2-8U^3-8U^2V+4UV^2-2U^3V)=0.\end{aligned}$$

The roots of this polynomial are

$$X2 = \frac{U^2+4U-2V}{V+2}, \tag{5.17}$$

$$X3 = \frac{U(V+4)}{2V+4-U}. \tag{5.18}$$

In order for these roots to exist we must have $(2V+4-U) \neq 0$. This presents us with three cases

(Case I) $$U = 2(V+2),$$

(Case II) $$X = X2, \text{ and}$$

(Case III) $$X = X3.$$

We continue to solve the system (5.6)-(5.11) one case at a time. In each case, we find ourselves factoring polynomials in several variables. Naturally, to see that our calculations are correct, one should take the product of the factors presented.

CASE I. In this case, $U = U1 = 2(V+2)$. Substituting this value of U into (5.8) yields

$$2 = \frac{(C+4)(V+4)}{2V+8} - \frac{C+2)(2V+6)}{X+2} = \frac{CX - 10C + 4X - 16 - 4VC - 8V}{2(X+2)}.$$

which gives us an expression for X as

$$X1 = \frac{24 + 10C + 4VC + 8V}{C}.$$

If we substitute $X = X1$ and $U = U1$ into (5.6) we find

$$2 = \frac{(C+2)(2V+6)}{V+2} - \frac{24 + 10C + 4VC + 8V}{2V+4}$$

which produces the expression for C given by

$$C1 = 2(V+2)$$

and so $C = U$. Now, if we substitute $C = U$ into (5.6) we get that $X = X2$ and hence we are reduced to Case II.

CASE II. In this case,we use the value $X = X2$ as given by (5.17). Substitute this value into C as given by (5.14) to get

$$C2 = \frac{-2U(U-V)}{U^2 + 2U - U^2 - 4U + 2V} = \frac{-2U(U-V)}{2(-U+V)} = U$$

Using $C = U$ and equation (5.10), we quickly find an expression for T as

$$T2 = \frac{UX}{V} = \frac{U(U^2 + 4U - 2V)}{V(V+2)}.$$

We now substitute $C = C2$, $T = T2$ and $X = X2$ into (5.11) and expand to get

$$\begin{aligned} 2 &= C\frac{X}{U} - (C-2)\frac{T-2}{X-2} \\ &= \frac{-2(8V+8UV-8U^2-12U^2V+U^4+2U^3-2V^3+4V^2-3U^3V+2U^2V^2+10UV^2-UV^3)}{U^2V^2+2U^2V+4UV^2+8UV-4V^3-12V^2-8V}. \end{aligned}$$

which simplifies to give us

$$\begin{aligned} 0 &= 3U^2V^2 - 10U^2V + 14UV^2 + 16UV - 6V^3 - 8V^2 - 8U^2 + U^4 + 2U^3 - 3U^3V - UV^3 \\ &= -(U-V)^2((U+6)V - (U^2+2U-8)). \end{aligned}$$

Since $U \neq V, -6$ by assumption,we have an expression for V given by

$$V2 = \frac{U^2+2U-8}{U+6}.$$

We substitute $V = V2$ into the values $X2$ and $T2$ to find $X = U + 4$ and $T = \frac{U(U+6)}{U-2}$. To conclude this case, we substitute $C = U$,$X = U + 4$, $T = \frac{U(U+6)}{U-2}$, and $V = \frac{U^2+2U-8}{U+6}$ into (5.9) and expand to find

$$\begin{aligned} 2 &= (C+4)\frac{V+4}{X+4} - (C+2)\frac{U+2}{T+2} \\ &= 2\frac{U^4+22U^3+156U^2+328U+64}{U^4+22U^3+156U^2+328U-192}. \end{aligned}$$

Therefore, Case II leads to the contradiction $64 = -192$.

CASE III. For this case, we use the value of $X = X3$ as given by (5.18).Substituting $X = X3$ into C as given by (5.14) we get an expression for C given by

$$C3 = \frac{2U(V-U)(2V+4-U)}{U(U+2)(2V+4-U) - U(V+4)(V+2)} = \frac{-2(U-2V-4)}{U-V-2}.$$

Similarly, substituting $X = X3$ into T as given by (5.15) and multiplying the numerator and denominator of this expression by $(2V+4-U)^2$ we get an expression for T given by

$$\begin{aligned} &T3 = \\ &\frac{(U^2-UV)(U(V+4))^2 + (2U^2-2UV)U(V+4)(2V+4-U)}{(-V-2)(U(V+4))^2 + (U+4+V)U^2(V+4)(2V+4-U) + (-U-2)UV(2V+4-U)^2} \\ &= \frac{-U^2(V+4)(U+4)(V+2)(V-U)}{U(V-U)(V+2)(2U^2-3UV-8V-16)} \\ &= \frac{-U(V+4)(U+4)}{2U^2-3UV-8V-16} \end{aligned}$$

Our next step is to substitute the values $X = X3, C = C3, T = T3$ into equation (5.10) and simplify to obtain a value of V in terms of U. We begin as follows

$$\begin{aligned}
2 &= \frac{1}{V}\frac{-2(U-2V-4)}{U-V-2}\frac{U(V+4)}{2V+4-U} \\
&\quad - \frac{1}{U-2}\left[\frac{-2(U-2V-4)}{U-V-2}-2\right]\left[\frac{-U(V+4)(U+4)}{2U^2-3UV-8V-16}-2\right] \\
&= \frac{1}{V}\frac{2U(V+4)}{U-V-2} \\
&\quad - \left[\frac{6V+12-4U}{(U-2)(U-V-2)}\right]\left[\frac{-U^2V+2UV-8U^2-16U+16V+32}{2U^2-3UV-8V-16}\right] = \\
&\frac{(U-V-2)(4U^3(V+4)-U^2(40V-6V^2)-U(96V-128+12V^2)+96V^2+192V)}{V(U-2)(U-V-2)(2U^2-3UV-8V-16)}.
\end{aligned}$$

Therefore,

$$2 = \frac{(4U^3V+16U^3-40U^2V+6U^2V^2-96UV+128U-12UV^2+96V^2+192V)}{V(U-2)(2U^2-3UV-8V-16)}$$

which simplifies to

$$\begin{aligned}
0 &= 16U^3-32U^2V-64UV-128U+16UV^2+64V^2+128V \\
&= (U-V)(16U^2-16UV-128-64V)
\end{aligned}$$

Since $U \neq V$,we have

$$16U^2-16UV-128-64V = 0$$

or in other words we have the value of V given by

$$V3 = \frac{U^2-8}{U+4}.$$

Now we substitute $V = V3$ into $C3$ to find $C = U$. As in Case I, $C = U$ can be substituted into (5.6) to get $X = X2$ and hence Case III also reduces to Case II. ∎

LEMMA 5.19. *$\mathcal{S}_k - \mathcal{S}_{k-1} = \mathcal{S}_1 - \mathcal{S}_0$ for all k and hence $\sigma(k,\ell) = \sigma(0,\ell)$ for all k and ℓ.*

PROOF: By Lemma 5.2, we may assume $C(k,\ell) \neq 0$ for all k and ℓ, and hence by (4.20) $\sigma(k,\ell) \neq 0$ for all k and ℓ. This allows us to use Lemma 4.27 and in particular (4.28). By Theorem 5.12, $\mathcal{S}_k - \mathcal{S}_{k-1} = \mathcal{S}_1 - \mathcal{S}_0$ for $k = -1, 0, 1, 2$. The result now follows easily from (4.28) by induction. ∎

LEMMA 5.20. *The A_1 representations $\mathcal{V}_k$ are equivalent for all k.*

PROOF: By the remarks in Example 1.11, it is sufficient to prove $\mathcal{S}_k = \mathcal{S}_0$ for all k. By Lemma 4.25 there is at most one ℓ_0 for which $C(-\ell_0, \ell_0) = 0$. Using (4.13),(4.20), and (4.21) one can conclude (see the proof of Lemma 5.2) that there is at most one ℓ_0 for which $\sigma(k,\ell_0) = 0$ for some k. Hence, (4.29) holds for all choices of ℓ except perhaps for $\ell = \ell_0, \ell_0 - 1$ where $\sigma(k,\ell_0) = 0$. By Lemma 5.19, (4.29) simplifies to

(5.21)

$$\begin{aligned}(C+2\ell) &= (s_{k+1}+\lambda+\ell)(s_{k+1}-\ell+1)\frac{C+2\ell}{X+2\ell} \\ &\quad - (s_{k+1}+\lambda+\ell+1)(s_{k+1}-\ell)\frac{C+2(\ell+1)}{X+2(\ell+1)} \\ &= (\mathcal{S}_{k+1}+(\lambda+\ell)(-\ell+1))\frac{C+2\ell}{X+2\ell} - (\mathcal{S}_{k+1}-\ell(\lambda+\ell+1)\frac{C+2(\ell+1)}{X+2(\ell+1)}\end{aligned}$$

where $X = \mathcal{S}_1 - \mathcal{S}_0 + \lambda$. If $X = C$ then this simplifies to give $C = \lambda$ and so $X = \lambda$ or equivalently $\mathcal{S}_1 - \mathcal{S}_0 = 0$ and the proof is completed by Lemma 5.19. If $X \neq C$ then (5.21) can be used to solve for $\mathcal{S}_{k+1}$. This gives $\mathcal{S}_{k+1}$ as an expression not involving k and hence the $\mathcal{S}_{k+1}$'s are all equal. ∎

THEOREM 5.22. *Let Λ be a weight of a pointed torsion free $A_1^{(1)}$ module*

$$\mathcal{V} = \sum_{k_0,k_1 \in \mathbf{Z}} \mathcal{V}_{\Lambda+k_0\alpha_0+k_1\alpha_1}.$$

If $\Lambda(d) = \mu$ then $\mathcal{V}$ is equivalent to $N(\mu,\vec{a})$ for some $\vec{a} = (a_1,a_2)$.

PROOF: Using the notation set down at the beginning of the Section, we write $\mathcal{V} = \sum_{k\in\mathbf{Z}} \mathcal{V}_k$. By Lemma 5.20 and Theorem 1.10 there exist complex numbers a_1 and a_2 such that for each k we can select a basis $\{u(k,\ell) \mid \ell \in \mathbf{Z}\}$ of $\mathcal{V}_k$ such that

$$\begin{aligned}(1\otimes x)u(k,\ell) &= (s-\ell)u(k,\ell+1), \\ (1\otimes y)u(k,\ell) &= (s+\lambda+\ell)u(k,\ell-1), \text{ and} \\ (1\otimes h)u(k,\ell) &= (\lambda+2l)u(k,\ell-1).\end{aligned}$$

where $s = a_2$ and $\Lambda(h_{\alpha_1}) = \lambda = a_1 - a_2$. This means that the s_k's of equations (4.1),(4,2), and (4.3) are all equal. By Lemma 4.22, $A(k,\ell) = s - \ell$ for all k and ℓ. By Lemma 4.23, $D(k,\ell) = \lambda + 2\ell$ for all k and ℓ. From this and Lemmas 4.24 and 4.25, $C(k,\ell)(\lambda+2\ell) = (C+2\ell)(\lambda+2\ell)$ and so $C(k,\ell) = (C+2\ell)$. We now can apply Lemma 4.19 to see that the value of $B(k,\ell)$ is not dependent on the value of k. Furthermore,

$$\begin{aligned}-2(s+\lambda+\ell)u(k,\ell-1) = -2(1\otimes y)u(k,\ell) &= [t^{-1}\otimes h, t\otimes y]u(k,\ell) \\ &= (B(k,\ell)D(k+1,\ell-1) - D(k,\ell)B(k-1,\ell))u(k,\ell-1).\end{aligned}$$

Hence, $-2(s+\lambda+\ell) = B(k,\ell)(D(k+1,\ell-1) - D(k,\ell)) = B(k,\ell)(-2)$ so that $B(k,\ell) = s+\lambda+\ell$.

Let $N(\mu,\vec{a}) = \text{lin. span}\{t^{\mu+k}x_1^{a_1+\ell}x_2^{a_2-\ell} \mid k,\ell \in \mathbf{Z}\}$. It now follows that the map

$$\psi : \mathcal{V} \longrightarrow N(\mu,\vec{a})$$

defined by linearity and $u(k,\ell) \mapsto t^{\mu+k}x_1^{a_1+\ell}x_2^{a_2-\ell}$ is an $A_1^{(1)}$ module isomorphism since ψ commutes with the operators in the generating set $\{1\otimes x, 1\otimes y, t^{-1}\otimes x, t\otimes y, d\}$ of $A_1^{(1)}$. ∎

6 Proof of Main Theorem - the A_{n-1} case. In this Section, we complete the proof of the Main Theorem. We assume that $n > 2$ and show that the $\hat{L}(A_{n-1})$ modules $N(\mu, \vec{a})$ constructed in Section 2 include all pointed torsion free $\hat{L}(A_{n-1})$ modules up to equivalence. Our general approach is a variation of that of Section 3 with Lemma 1.13 and Theorem 5.22 playing the role of Corollary 1.12.

We first identify A_{n-1} with $1 \otimes A_{n-1} \subset \hat{L}(A_{n-1})$. Let $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ denote the simple roots of $\hat{L}(A_{n-1})$ with $\{\alpha_1, \ldots, \alpha_{n-1}\}$ the simple roots of A_{n-1}. For each positive root α of A_{n-1}, let $A_1(\alpha)$ be the A_1 subalgebra of A_{n-1} generated by x_α, $x_{-\alpha}$, and h_α. Assume the elements x_α, $x_{-\alpha}$, and h_α have been normalized so that they have the same structure constants as the basis x, y and h of Example 1.11. Let $\hat{L}(A_1(\alpha))$ be the corresponding $A_1^{(1)}$ subalgebra of $\hat{L}(A_{n-1})$.

Let $\mathcal{V}$ be a pointed torsion free $\hat{L}(A_{n-1})$ module and let Λ be a weight of $\mathcal{V}$. Without loss of generality we may assume $\Lambda(h_{\alpha_1}) \neq 0$. $\mathcal{V}$ admits a weight space decompostion

$$\mathcal{V} = \sum_{k_i \in \mathbf{Z}} \mathcal{V}_{\Lambda + k_0\alpha_0 + \cdots + k_{n-1}\alpha_{n-1}} \tag{6.1}$$

where each weight space is 1-dimensional. For any fixed integer k the space

$$\mathcal{V}_k = \sum_{k_1, \ldots, k_n \in \mathbf{Z}} \mathcal{V}_{\Lambda + k\alpha_0 + k_1\alpha_1 + \cdots + k_n\alpha_n}$$

is a pointed torsion free A_{n-1} module.

LEMMA 6.2. *$\mathcal{V}_k$ and $\mathcal{V}_0$ are equivalent A_{n-1} modules for all $k \in \mathbf{Z}$.*

PROOF: Let λ equal Λ restricted to $1 \otimes H$ the Cartan subalgebra of the A_{n-1} algebra $1 \otimes A_{n-1}$. Then λ is a weight common to each $\mathcal{V}_k$ with $\lambda(h_{\alpha_1}) \neq 0$. Let v_λ and u_λ be weight vectors of weight λ in $\mathcal{V}_k$ and $\mathcal{V}_0$, respectively. Let the universal enveloping algebra of $\hat{L}(A_1(\alpha))$ act on v_λ to generate a pointed torsion free $\hat{L}(A_1(\alpha))$ module. The vector u_λ is in this module because $u_\lambda \in \mathbf{C}((t \otimes h_\alpha)^k)v_\lambda$ or $u_\lambda \in \mathbf{C}((t^{-1} \otimes h_\alpha)^{|k|})v_\lambda$ accordingly as $k \geq 0$ or $k < 0$. Set $\alpha = \alpha_1$, α_2, and $\alpha_1 + \alpha_2$. According to Theorem 5.22 we have

$$x_{-\alpha_2} x_{\alpha_2} v_\lambda = g_1 v_\lambda,\ x_{-\alpha_2} x_{\alpha_2} u_\lambda = f_1 v_\lambda \Longrightarrow g_1 = f_1, \text{ and}$$
$$x_{-(\alpha_1+\alpha_2)} x_{\alpha_+\alpha_2} v_\lambda = g_2 v_\lambda,\ x_{-(\alpha_1+\alpha_2)} x_{\alpha_1+\alpha_2} u_\lambda = f_2 u_\lambda \Longrightarrow g_2 = f_2.$$

The result now follows from Lemma 1.13. ∎

THEOREM 6.3. *Let $\mathcal{V} = \sum_{k_i \in \mathbf{Z}} \mathcal{V}_{\Lambda + k_0\alpha_0 + \cdots + k_{n-1}\alpha_{n-1}}$ be a pointed torsion free $\hat{L}(A_{n-1})$ module. Let $N(\vec{a})$ be the pointed torsion free A_{n-1} module given by (1.9) which is equivalent to $\mathcal{V}_0 = \sum_{k_1, \ldots, k_{n-1} \in \mathbf{Z}} \mathcal{V}_{\Lambda + k_1\alpha_1 + \cdots + k_{n-1}\alpha_{n-1}}$ and let $\mu = \Lambda(d)$. Then $\mathcal{V}$ is equivalent to the pointed torsion free $\hat{L}(A_{n-1})$ module $N(\mu, \vec{a})$.*

PROOF: Let $\theta = \alpha_1 + \cdots + \alpha_{n-1} = \epsilon_1 - \epsilon_n$. Let $\{w(k, \ell_1, \ldots, \ell_{n-1}) \mid k, \ell_1, \ldots, \ell_{n-1} \in \mathbf{Z}\}$ be a basis of $\mathcal{V}$ such that $(t^{-1} \otimes x_{-\theta})w(-\ell, \ell, \ldots, \ell) = (a_n - \ell)w(-\ell - 1, \ell + 1, \ldots, \ell + 1)$ for all $\ell \in \mathbf{Z}$ and such that the linear map $\psi : \mathcal{V} \longrightarrow N(\mu, \vec{a})$ defined by

$$w(k, \ell_1, \ldots, \ell_{n-1}) \mapsto t^{\mu+k} x_1^{a_1+\ell_1} x_2^{a_2+\ell_2-\ell_1} \cdots x_{n-1}^{a_{n-1}+\ell_{n-1}-\ell_{n-2}} x_n^{a_n-\ell_{n-1}}$$

is an A_{n-1} module isomorphism where the action of A_{n-1} on $\mathcal{V}$ is that inherited from $1 \otimes A^{n-1}$. This map exists by Lemma 6.2.

The result follows once we have shown that $t \otimes x_{-\theta}$ and $t^{-1} \otimes x_\theta$ commute with ψ since $\hat{L}(A_{n-1})$ is generated by $(1 \otimes A(n-1)) \cup \{t \otimes x_{-\theta}, t^{-1} \otimes x_\theta\}$.

Let the universal enveloping algebra of $\hat{L}(A_1(\theta))$ act on $w(0,0,\ell_2,\ldots,\ell_{n-1})$ to generate

$$(6.4) \qquad \mathcal{V}_{\ell_2,\ldots,\ell_{n-1}} = \text{lin. span}\{u(k,\ell) = w(k,\ell,\ell_2+\ell,\ldots,\ell_{n-1}+\ell) \mid k,\ell \in \mathbf{Z}\}$$

a pointed torsion free $A_1(\theta)$ module.

Let $x = x_\theta$, $y = x_{-\theta}$ and $h = h_\theta$ and set $\lambda = (\Lambda + \ell_2\alpha_2 + \cdots + \ell_{n-1}\alpha_{n-1})(h_\theta)$. We can now use equations (4.5)-(4.8) to define $A(k,\ell)$, $B(\ell,k)$, $C(k,\ell)$, and $D(k,\ell)$. By our choice of basis we have that all the s_k's in Lemma 4.10 are equal to the fixed value $s = a_n - \ell_{n-1}$. Showing that $t \otimes x_{-\theta}$ and $t^{-1} \otimes x_\theta$ commute with ψ is equivalent to showing that $A(k,\ell) = s - \ell$ and $B(k,\ell) = s + \lambda + 1$.

First we consider the module given by (6.4) when $\ell_2 = \cdots = \ell_{n-1} = 0$. In this case we know that $A(-\ell,\ell) = s - \ell$ by choice of basis and hence by induction and (4.11) $A(k,\ell) = s - \ell$. The remainder of the argument in this case follows verbatum the argument of Theorem 5.22. This gives us that

$$\begin{aligned}(t \otimes x_{-\theta})\psi(w(k,\ell,\ldots,\ell)) &= \psi((t \otimes x_{-\theta})w(k,\ell,\ldots,\ell)) \\ (t^{-1} \otimes x_\theta)\psi(w(k,\ell,\ldots,\ell)) &= \psi((t^{-1} \otimes x_\theta)w(k,\ell,\ldots,\ell))\end{aligned}$$

for all k and ℓ.

Now let $\ell_1,\ldots,\ell_{n-1}$ be arbitrary and let

$$\ell_{\min} = \min\{\ell_1,\ldots,\ell_{n-1}\} \text{ and } \ell_{\max} = \max\{\ell_1,\ldots,\ell_{n-1}\}.$$

Since $\overset{0}{N}(\vec{a})$ is isomorphic to $\sum_{\ell_1,\ldots,\ell_{n-1} \in \mathbf{Z}} \mathcal{V}_{\Lambda+\ell_1\alpha_1+\cdots+\ell_{n-1}\alpha_{n-1}}$ under the map ψ and x_θ commutes with x_{α_i} for $i = 1,\ldots,n-1$, we know that there is some complex number κ such that

$$\begin{aligned}&(1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}(t^{-1} \otimes x_\theta)\psi(w(k,\ell_1,\ell_2,\ldots,\ell_{n-1})) \\ &= (t^{-1} \otimes x_\theta)(1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}\psi(w(k,\ell_1,\ldots,\ell_{n-1})) \\ &= (t^{-1} \otimes x_\theta)\psi((1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}w(k,\ell_1,\ldots,\ell_{n-1})) \\ &= (t^{-1} \otimes x_\theta)\psi(\kappa w(k,\ell_{\max},\ldots,\ell_{\max})) \\ &= \psi((t^{-1} \otimes x_\theta)\kappa w(k,\ell_{\max},\ldots,\ell_{\max})) \\ &= \psi((t^{-1} \otimes x_\theta)(1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}w(k,\ell_1,\ldots,\ell_{n-1})) \\ &= \psi((1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}(t^{-1} \otimes x_\theta)w(k,\ell_1,\ldots,\ell_{n-1})) \\ &= (1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}\psi((t^{-1} \otimes x_\theta)w(k,\ell_1,\ldots,\ell_{n-1})).\end{aligned}$$

Naturally, the torsion free property allows us to cancel

$$(1 \otimes x_{\alpha_1})^{\ell_{\max}-\ell_1} \cdots (1 \otimes x_{\alpha_{n-1}})^{\ell_{\max}-\ell_{n-1}}$$

from the first and last expressions to give us that

$$(t^{-1}\otimes x_\theta)\psi(w(k,\ell_1,\ldots,\ell_{n-1}))=\psi((t^{-1}\otimes x_\theta)w(k,\ell_1,\ldots,\ell_{n-1})).$$

In a similar manner, one can argue that

$$\begin{aligned}&(1\otimes x_{-\alpha_1})^{\ell_1-\ell_{\min}}\cdots(1\otimes x_{-\alpha_{n-1}})^{\ell_{n-1}-\ell_{\min}}(t\otimes x_{-\theta})\psi(w(k,\ell_1,\ell_2,\ldots,\ell_{n-1}))\\ &=(1\otimes x_{-\alpha_1})^{\ell_1-\ell_{\min}}\cdots(1\otimes x_{-\alpha_{n-1}})^{\ell_{n-1}-\ell_{\min}}\psi((t\otimes x_{-\theta})w(k,\ell_1,\ldots,\ell_{n-1})).\end{aligned}$$

and so we get

$$(t\otimes x_{-\theta})\psi(w(k,\ell_1,\ldots,\ell_{n-1}))=\psi((t\otimes x_{-\theta})w(k,\ell_1,\ldots,\ell_{n-1})).$$

This completes the proof. ∎

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References

1. D. J. Britten and F. W. Lemire, *A classification of simple Lie modules having a 1-dimensional weight space*, Trans. Amer. Math. Soc. **299** (1987), 683–697.
2. D. J. Britten and F. W. Lemire, *Irreducible representations of A_n with a 1-dimensional weight space*, Trans. Amer. Math. Soc. **273** (1982), 509–540.
3. Suren L. Fernando, *Simple weight modules of complex Lie algebras*, Ph. D. Thesis, Univ. of Wisconsin, 1983.
4. J. Humphreys, *Introduction to Lie algebras and representations theory, No. 9, Springer-Verlag, New York, 1972.*
5. Victor G. Kac, *Infinite Dimensioal Lie Algebras*, Progress in Mathematics, Vol. 44, Birkhauser, Boston 1983.
6. F. W. Lemire, *A new family of irreducible representations of A_n*, Canad. Math. Bull. **19** (1975), 1161–1164.

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